Indexed Grammars (Aho)

$$G = (V_T, V_N, F, P, S)$$

V_T - terminal symbols
V_N - nonterminal symbols
S - start symbol
F is a finite set of special ("indexed") productions of the form:

$$A \longrightarrow \alpha,$$

where $A \in V_N$, $\alpha \in (V_T \cup V_N)^*$
P is a finite set of productions of the form

$$A \longrightarrow X_1\psi_1\ X_2\psi_2\ \ldots\ X_m\psi_m$$

where $A \in V_N$

$$X_1, X_2, \ldots, X_m \in (V_T \cup V_N)$$

$$\psi_1, \psi_2, \ldots, \psi_m \in F^*$$

(they are strings of indexed productions)
If $X_i \in V_T$, then $\psi_i = e$.

Derivation in Indexed Grammar:

1. **If**

$\alpha A\theta\beta$ is a string such that
α and $\beta \in (V_N F^* \cup V_T)^*$
$A \in V_N$, $\theta \in F^*$, and
$A \longrightarrow X_1\psi_1\ X_2\psi_2\ \ldots\ X_m\psi_m$ is <u>a production from P,</u>
where $X_i \in (V_T \cup V_N)$, $\psi_i \in F^*$ for every i,

then

$\alpha A\theta\beta \Rightarrow \alpha(X_1\varphi_1\ X_2\varphi_2\ \ldots\ X_m\varphi_m)\beta$
Here $\varphi_i = e$, if $X_i \in V_T$;
$\varphi_i = \psi_i\theta$, if $X_i \in V_N$.

2. **If**

$\alpha A\mathbf{f}\theta\beta$ is a string such that

α and $\beta \in (V_{\mathbf{N}}F^* \cup V_{\mathbf{T}})^*$

$A \in V_{\mathbf{N}}$,

$\mathbf{f} \in F$

$\theta \in F^*$, and

$A \longrightarrow X_{\mathbf{1}}X_{\mathbf{2}} \ldots X_{\mathbf{m}}$ is an <u>indexed production</u> from **f**

then

$\alpha A f\theta\beta \Rightarrow \alpha(X_{\mathbf{1}}\varphi_{\mathbf{1}}\ X_{\mathbf{2}}\varphi_{\mathbf{2}} \ldots X_{\mathbf{m}}\varphi_{\mathbf{m}})\beta$.

Here $\varphi_{\mathbf{i}} = e$ (empty string), if $X_{\mathbf{i}} \in V_{\mathbf{T}}$;

$\varphi_{\mathbf{i}} = \theta$, if $X_{\mathbf{i}} \in V_{\mathbf{N}}$.

<u>**f** disappears</u>

Indexed Grammars (Aho)

1. Let L be the same language

$L = \{a^{n}b^{n}c^{n}, n \geq 1\}$

$G = (V_T, V_N, F, P, S)$,

$V_T = \{a, b, c\}$,

$V_N = \{S, W, A, B, C\}$

$F = \{f, g\}$

P constists of 3 productions:

1. S —> Wg
2. W —> Wf
3. W —> ABC, where indices are

   f = {A —> aA, B —> bB, C —> cC}

   g = {A —> a, B—> b, C —> c}

Show the derivation of the string $a^{n}b^{n}c^{n}$.

Derivation for $a^2b^2c^2$:

S => Wg => Wfg => Afg Bfg Cfg =>. . .aAg bBg cCg => . . . => aabbcc

for $a^{n}b^{n}c^{n}$:

$S \underset{1}{\Rightarrow} Wg$

$\mathbf{n-1} \left\{ \begin{array}{l} \underset{2}{\Rightarrow} Wfg \\ \underset{2}{\Rightarrow} Wffg \\ \ldots\ldots\ldots \\ \underset{2}{\Rightarrow} Wf^{n-1}g \end{array} \right.$

$\underset{3}{\Rightarrow}$ $Af^{n-1}g\ Bf^{n-1}g\ Cf^{n-1}g$

$n-1 \left\{ \begin{array}{l} \underset{f\ (3\ times)}{\Rightarrow}\ aAf^{n-2}g\ bBf^{n-2}g\ cCf^{n-2}g \\ \quad \cdots \\ \underset{f\ (3\ times)}{\Rightarrow}\ a^{n-1}Ag\ b^{n-1}Bg\ c^{n-1}Cg \end{array} \right.$

$\underset{g\ (3\ times)}{\Rightarrow}$ $a^n b^n c^n$

Total = 4n+2 steps

2. $G = (V_T, V_N, F, P, S)$

$V_T = \{a, b\}$

$V_N = \{S, T, A, B, C\}$

$F = \{f, g\}$

P:

1. S —> Tf
2. T —> Tg
3. T —> ABA,

f = {A —> a, B —> b, C —> b}

g = {A —> aA, B —> bBCC, C —> bC}

Show that G generates the language $\{a^n b^{n^2} a^n, n \geq 1\}$.

Derivation for $a^2b^4a^2$

$$S \overset{1}{\Rightarrow} Tf \overset{2}{\Rightarrow} Tgf \overset{3}{\Rightarrow} Agf\ Bgf\ Agf$$

$$\overset{g_1}{\Rightarrow} aAf\ Bgf\ Agf \overset{f_1}{\Rightarrow} a^2BgfAgf$$

$$\overset{g_2}{\Rightarrow} a^2bBfCfCfAgf \overset{f_2}{\Rightarrow} a^2b^2CfCfAgf \overset{f_3}{\Rightarrow} a^2b^3CfAgf$$

$$\overset{f_3}{\Rightarrow} a^2b^4Agf \overset{g_1}{\Rightarrow} a^2b^4aAf \overset{f_1}{\Rightarrow} a^2b^4a^2$$

3. $L = \{0^{n^2}, n \geq 1\}$ $V_T = \{0\}$, $V_N = \{S, A, B, C, D\}$

F = {f, g}, P consists of 4 productions:

1. S —> Ag
2. A —> Af
3. A —> B
4. B —> CD

where indices are

f = {D —> B, C —> 00C}

g = {D —> e , C —> 0}

Show the derivation of the string 0^{n^2}.

$$S \underset{1}{\Rightarrow} Ag \underbrace{\underset{2}{\Rightarrow} Afg \underset{2}{\Rightarrow} \ldots \underset{2}{\Rightarrow}}_{n-1} Af^{n-1}g \underset{3}{\Rightarrow} \underline{B}f^{n-1}g \Rightarrow$$

$n-1$ times:

$$\underset{4}{\Rightarrow} Cf^{n-1}g\ \underline{Df^{n-1}}g \quad \underset{f}{\Rightarrow} Cf^{n-1}g\ \underline{B}f^{n-2}g$$

$$\underset{4}{\Rightarrow} Cf^{n-1}g\ Cf^{n-2}g\ Df^{n-2}g \underset{f}{\Rightarrow} Cf^{n-1}g\ Cf^{n-2}gBf^{n-3}g$$

. . . . . .

$$\Rightarrow Cf^{n-1}g\ Cf^{n-2}g \ldots CfgCgDg$$

$$\underset{4}{}$$

$$\underset{g}{\Rightarrow} \underline{Cf^{n-1}g}\ \underline{Cf^{n-2}g}\ \ldots \underline{Cfg}\ \underline{Cg}$$

$n-1$ times:

$$\underset{f}{\Rightarrow} 00Cf^{n-2}gCf^{n-2}g\ldots CfgCg$$

. . . .

$$\underset{f}{\Rightarrow} 0^{2(n-1)}\underline{Cg}Cf^{n-2}g\ldots CfgCg$$

$=> 0^{2n-1}Cf^{n-2}g\ \ . . . Cfg\ Cg => . . .=> 0^{2n-1}0^{2n-3}. . .\ 0^3\ 0^1 = 0^{n^2}$

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$$(2n-1) + (2n-3) + . . . + 3 + 1 = n^2$$

4. Prove that the following indexed grammar generates this language:
$\{0^n \mid n$ is a power of $2\}$.

$G = \{V_T, V_N, F, P, S\}$
$V_T = \{\mathbf{0}\}, V_N = \{S, A, B\}$
$F = \{\mathbf{a}, \mathbf{b}\}$
P: 1. $S \longrightarrow A\mathbf{b}$
2. $A \longrightarrow A\mathbf{a}$
$\mathbf{a} = \{A \longrightarrow BB, B \longrightarrow BB\}$
$\mathbf{b} = \{A \longrightarrow \mathbf{0}, B \longrightarrow \mathbf{0}\}$

Derivation

n = 0 $\quad S \overset{1}{=>} Ab \overset{b_1}{=>} 0$

n = 1 $\quad S \overset{1}{=>} Ab \overset{2}{=>} Aab \overset{a_1}{=>} BbBb \overset{b_2 \times 2}{=>} 0^2$

n > 1 $\quad S \overset{1}{=>} Ab$

n times:

$\overset{2}{=>} Aab$

$\overset{2}{=>} Aa^2b$

…….

$\overset{2}{=>} Aa^nb$

n times:

$\mathbf{a_1} \Rightarrow Ba^{n-1}bBa^{n-1}b$

$\mathbf{a_2} \times 2 \Rightarrow \underline{Ba^{n-2}bB\,a^{n-2}b}\ \underline{Ba^{n-2}bB\,a^{n-2}b}$

$\mathbf{a_2} \times 2^{\mathbf{2}} \Rightarrow \ldots.$

$\ldots\ldots.$

$\mathbf{a_2} \times 2^{\mathbf{n-1}} \Rightarrow (Bb)2^{\mathbf{n}}$

2^n times:

$\mathbf{b_2} \Rightarrow 0(Bb)2^{\mathbf{n}}\text{-}1$

$\ldots\ldots\ldots$

$\mathbf{b_2} \Rightarrow \mathbf{0}2^{\mathbf{n}}$